

Semiclassical Theory of Linear Magnetoresistance in Crystalline Conductors with Broken Time-Reversal Symmetry

Hua Chen,¹ Yang Gao,¹ Di Xiao,² Allan H. MacDonald,¹ and Qian Niu¹

¹*Department of Physics, University of Texas at Austin, Austin, TX 78712, USA*

²*Department of Physics, Carnegie Mellon University, Pittsburgh, PA 15213, USA*

Onsager relations permit linear response of the resistivity to an external magnetic field only when time reversal symmetry is broken. By employing semiclassical Boltzmann transport theory generalized to account for anomalous velocities and orbital and spin magnetic moments, we obtain an explicit expression for the linear magnetoresistance of a crystalline conductor. Like the anomalous Hall effect, also permitted only when time-reversal symmetry is broken, it includes both intrinsic and extrinsic contributions. The intrinsic linear magnetoresistance, which is dominant in strongly disordered systems, is due to the influence on transport of momentum-space Berry curvatures and momentum-dependent Bloch state magnetic moments. For the case of surface state transport in magnetically ordered topological insulators we predict positive magneto-resistance that is linear in field but independent of field direction.

PACS numbers: 72.15.Gd, 73.43.Qt, 75.47.-m

The dependence of the longitudinal resistance of a conductor on an external magnetic field is defined as its magnetoresistance, and is one of the central quantities measured in magnetotransport experiments. Distinct magnetoresistance characteristics are associated with superconductivity, quantum Hall effect, giant magnetoresistance, and other key phenomena of condensed matter physics. Linear magnetoresistance (LMR), first observed in simple metals such as potassium and sodium in the 1960s [1–3], has long been the subject of intense study and lively debate. Recently interest in LMR has been revived because of its appearance in many new materials, including silver chalcogenides [4], doped InSb [5, 6], graphene multilayers [7], topological insulators [8–11], Dirac [12–16] and Weyl [17, 18] semimetals. Both extreme quantum limit physics [19–25], and inhomogeneity [26–30] have been proposed as potential explanations.

It is strictly speaking meaningful to talk about linearity only in the limit that the magnetic field \mathbf{B} is weak, so that an expansion of transport coefficients in powers of the magnetic field strength B is well defined. In this weak field regime Onsager relations forbid any odd powers of B in the series expansion of the longitudinal conductivity σ_{xx} when only the magnetic field breaks time reversal symmetry [31]. If time-reversal symmetry is broken intrinsically, however, as in the case of a ferromagnet with magnetization \mathbf{M} , Onsager relations require only that $\sigma_{xx}(-\mathbf{B}, -\mathbf{M}) = \sigma_{xx}(\mathbf{B}, \mathbf{M})$ and LMR is allowed.

Although the appearance of LMR in magnetic systems is natural from a symmetry point of view, a quantitative and comprehensive theory has, to our knowledge, been absent. The goal of this Letter is to provide one. We are motivated in part by recent progress [32] in understanding the mechanisms that contribute to another important transport phenomenon in magnetic materials, the anomalous Hall effect (AHE). In AHE theory, the semiclassical Boltzmann transport theory has proven to be a

useful tool that can not only provide an intuitive physical picture, but also quantitative predictions [33, 44]. In this Letter, we extend the semiclassical Boltzmann theory to magnetic materials that are subject to *both* electric and magnetic fields, and show that there are close connections between the AHE and LMR. Specifically we find, as summarized in Fig. 1 (a), that as in the AHE case there are both intrinsic (Berry curvature, and spin and orbital moment) and extrinsic (skew scattering and side jump scattering) contributions to LMR. The former are weakly dependent on scattering and are important at high impurity densities and at high temperatures, while the latter are dominant in the clean limit or at low temperatures. Our theory is valid in the low magnetic field regime where the product of the cyclotron frequency ω_c and the transport relaxation time τ is much smaller than 1, and is complementary to the theories which assume the extreme quantum limit [19–21, 24]. We demonstrate that intrinsic LMR can be very large and have a highly non-trivial dependence on Fermi energy in systems with strong spin-orbit coupling, and that this can lead to exotic behavior like that associated with surface state transport in magnetically doped topological insulators.

Semiclassical Boltzmann transport theory consists of three main components: i) the Boltzmann equation, ii) Bloch state wave-packet equations of motion, and iii) the semiclassical description of scattering. When a non-magnetic homogeneous system is subject to both electric and magnetic fields, the steady-state Bloch state distribution function $f_{n\mathbf{k}}$ is determined by the three corresponding equations:

$$\dot{\mathbf{k}} \cdot \frac{\partial f_{n\mathbf{k}}}{\partial \mathbf{k}} = \left(\frac{df_{n\mathbf{k}}}{dt} \right)_{\text{scatt}}, \quad (1)$$

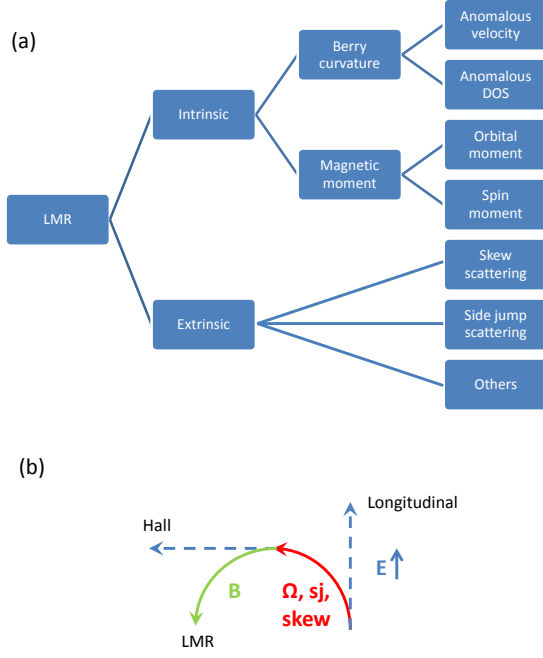


FIG. 1: (Color online) (a) Classification of mechanisms contributing to LMR in crystalline conductors with broken time-reversal symmetry. (b) Relationship between microscopic processes leading to LMR and those to the AHE (Berry curvature, side-jump scattering, and skew scattering).

where n and \mathbf{k} are band and momentum labels;

$$\dot{\mathbf{r}} = \frac{1}{\hbar} \frac{\partial \epsilon_{n\mathbf{k}}}{\partial \mathbf{k}}, \quad (2a)$$

$$\dot{\mathbf{k}} = \frac{e}{\hbar} \mathbf{E} + \frac{e}{\hbar} \dot{\mathbf{r}} \times \mathbf{B}, \quad (2b)$$

where $e = -|e|$ is the electronic charge [36];

$$-\left(\frac{df_{n\mathbf{k}}}{dt}\right)_{\text{scatt}} = \sum_{n'\mathbf{k}'} W_{n\mathbf{k},n'\mathbf{k}'} f_{n\mathbf{k}} - W_{n'\mathbf{k}',n\mathbf{k}} f_{n'\mathbf{k}'}, \quad (3)$$

where W is the quantum mechanical scattering rate [35] from state $n\mathbf{k}$ to state $n'\mathbf{k}'$. Once $f_{n\mathbf{k}}$ is determined by solving the Boltzmann equation, the electric current density is

$$\mathbf{J} = \sum_n \int \frac{d^N k}{(2\pi)^N} e \dot{\mathbf{r}} f_{n\mathbf{k}}, \quad (4)$$

where N is the momentum-space dimension. The conductivity tensor is defined by $\hat{\sigma}_{ij} = \partial \mathbf{J}_i / \partial \mathbf{E}_j$ evaluated at $\mathbf{E} = 0$ and the resistivity tensor is $\hat{\rho} = \hat{\sigma}^{-1}$.

Now we discuss how semiclassical Boltzmann theory is modified in the case of magnetically ordered systems. First of all the electronic structure of the system is modified by the coupling of the external fields to (1) Berry curvature $\Omega_{n\mathbf{k}} \equiv i \langle \nabla_{\mathbf{k}} u_{n\mathbf{k}} | \times | \nabla_{\mathbf{k}} u_{n\mathbf{k}} \rangle$ and (2) orbital and spin magnetic moments (\mathbf{m}_s and \mathbf{m}_o , respectively). The former gives rise to the “anomalous velocity” term $-\dot{\mathbf{k}} \times \Omega_{n\mathbf{k}}$ on the right hand side of Eq. 2a [33] and modifies the momentum space volume element $d^N k / (2\pi)^N$ in Eq. 4 by the factor $D_{n\mathbf{k}} \equiv 1 - (e/\hbar) \mathbf{B} \cdot \Omega_{n\mathbf{k}}$ [38], while the latter is responsible for Zeeman interactions which modify the argument of the *equilibrium* distribution function $f_0(\epsilon_{n\mathbf{k}})$ [42] and give a corresponding correction to the group velocity $\partial \epsilon_{n\mathbf{k}} / \partial \hbar \mathbf{k}$. In addition to modifying the perfect crystal electronic structure, time-reversal symmetry breaking induces an antisymmetric contribution to the scattering rate matrix W in Eq. 3, referred to as skew scattering [44]; it also leads to a coordinate shift $\delta \mathbf{r}_{n'\mathbf{k}',n\mathbf{k}}$ (side jump) between incoming and outgoing Bloch states in scattering events [37, 44]. Averaged over time, side-jump scattering gives rise to a new velocity term $\mathbf{v}^{sj} \equiv \sum_{n'\mathbf{k}'} W_{n\mathbf{k},n'\mathbf{k}'} \delta \mathbf{r}_{n'\mathbf{k}',n\mathbf{k}}$ to be added to $\dot{\mathbf{r}}$ in Eq. 4. To account for the change in band energy when this coordinate shift is coupled to an electric field, a correction $(\partial f_0 / \partial \epsilon_{n\mathbf{k}}) e \mathbf{E} \cdot \delta \mathbf{r}_{n'\mathbf{k}',n\mathbf{k}}$ needs to be added to $f_{n'\mathbf{k}'}$ in Eq. 3 [44].

Eqs. 1-4 together with the modifications listed above complete the semiclassical Boltzmann theory of magnetotransport in magnetically ordered systems. We have ignored only the potential magnetic field dependence of the scattering rates, which in a relaxation time approximation will make the scattering time τ depend on B . This correction requires knowledge of scattering processes and a fully quantum mechanical scattering theory [39–41], and is beyond the scope of this paper. (In Fig. 1(a) we have categorized this contribution to LMR under “Others”.) For illustrative purposes we ignore anisotropy of the Fermi surface and the scattering potential. The Boltzmann equation can then be solved analytically [45] (details are included in [42]). We obtain for the correction to the longitudinal conductance linear in B :

$$\begin{aligned}
\sigma_{xx}^{(1)} &= \sigma_{intr}^{(1)} + \sigma_{extr}^{(1)} \\
&= -e^2 \tau^{\parallel} B \sum_n \int \frac{d^N k}{(2\pi)^N} \left\{ \frac{\partial f_0}{\partial \epsilon_{n\mathbf{k}}} \left[\frac{e}{\hbar} \Omega_{n\mathbf{k}}^z (v_x^0)^2 - \frac{2}{\hbar} \frac{\partial m_{n\mathbf{k}}^z}{\partial k_x} v_x^0 \right] - \frac{\partial^2 f_0}{\partial \epsilon_{n\mathbf{k}}^2} \left[m_{n\mathbf{k}}^z + \frac{\delta\mu(B)}{B} \right] (v_x^0)^2 \right\} \\
&\quad + e^3 (\tau^{\parallel})^2 B \sum_n \int \frac{d^N k}{(2\pi)^N} \frac{\partial f_0}{\partial \epsilon_{n\mathbf{k}}} \alpha_{xx} \left[2 \frac{\tau^{\parallel}}{\tau^{\perp}} (v_x^0)^2 + v_x^0 v_y^{sj} \right],
\end{aligned} \tag{5}$$

where we have assumed that $\mathbf{B} = B\hat{z}$. In Eq. 5 τ^{\parallel} is the ordinary transport relaxation time, τ^{\perp} is the skew-scattering-induced transverse relaxation time, $v_x^0 = \partial_{k_x} \epsilon / \hbar$ is the x component of the group velocity, $m_{n\mathbf{k}}^z$ is the z component of the total magnetic moment including both spin and orbital contributions, and $\alpha_{xx} = \partial_{k_x}^2 \epsilon / \hbar^2$ is the inverse effective mass. For normal materials the anomalous Hall conductance, *i.e.* $\sigma_{xy}^{(0)}$, is much smaller than the longitudinal conductance. It follows that the LMR is given by

$$\text{LMR} \equiv \frac{\rho_{xx}^{(1)}}{\rho_{xx}^{(0)}} = \frac{-\sigma_{xx}^{(1)}}{\sigma_{xx}^{(0)}}. \tag{6}$$

Several comments are in order concerning these results:

(i) $\sigma_{intr}^{(1)}$ is proportional to the transport relaxation time τ^{\parallel} , while $\sigma_{extr}^{(1)}$ is proportional to $(\tau^{\parallel})^2$. Therefore the intrinsic LMR contribution is independent of (τ^{\parallel}) and relatively more important in more disordered systems with smaller τ . In contrast, the contribution due to skew scattering (τ^{\perp} in Eq. 5) and side jump scattering (v^{sj}) will be dominant when there are few impurities, or τ^{\parallel} is large. These differences between the intrinsic and the extrinsic contributions are similar to those in AHE theory [32, 44].
(ii) Several contributions to Eq. 5 are closely related to the AHE. For example, the $\sigma_{intr}^{(1)}$ contribution whose integrand is proportional to $\Omega_{n\mathbf{k}}^z$ can be traced back to the cross product between the anomalous velocity $\mathbf{v}^a \equiv -\frac{e}{\hbar} \mathbf{E} \times \boldsymbol{\Omega}_{n\mathbf{k}}$ and \mathbf{B} in the generalized form of Eq. 2b. This term provides an additional force along or opposite to the direction of \mathbf{E} when $\mathbf{E} \perp \mathbf{B}$. Similarly, the integrand in $\sigma_{extr}^{(1)}$ is proportional to $\omega_c \tau^{\parallel}$ times corrections to the nonequilibrium distribution function due to skew scattering and side jump scattering. In fact, the relation between the microscopic processes leading to LMR and those to the AHE can be understood through the simple picture illustrated in Fig. 1 (b). Consider two consecutive microscopic processes that can individually give rise to a transverse component of the motion of a charge carrier. One of them corresponds to the mechanisms that can lead to the AHE, such as anomalous velocity, side jump scattering, and skew scattering, and the other is due to the Lorentz force. These two effects in combination result in an extra longitudinal component of the charge carrier's motion, and hence modify the lon-

gitudinal conductance or resistance.

(iii) The contributions due to orbital and spin magnetic moments do not have an intuitive physical picture. However, their nontrivial dependence on \mathbf{k} indicates that they are all related to spin-orbit coupling, suggesting that large spin-orbit coupling is favorable for the appearance of intrinsic LMR in magnetic systems. Alternatively, \mathbf{k} -dependent magnetic moments can also arise from non-collinear magnetic order.

(iv) The correction to the density of states associated with momentum-space Berry curvature ($D_{n\mathbf{k}}$) will also lead to change of the total volume (number of states) under the Fermi surface [33] at a given Fermi energy. Assuming the total number of electrons is conserved, such a modification will change the chemical potential and hence contribute to σ_{xx} [“Anomalous DOS” in Fig. 1 (a) and the $\delta\mu$ term in Eq. 5]. We will discuss this contribution in more detail below.

Since our theory is valid in the semiclassical regime where $\omega_c \tau \ll 1$, the LMR predicted here is expected to persist to higher fields in more disordered systems where τ is small. In this regime the intrinsic contribution to LMR will be dominant. Below we use a concrete model to shed light on the different mechanisms included in the intrinsic contribution, and to discuss interesting effects that can be measured experimentally. For these purposes we study LMR in the two-dimensional gapped Dirac model,

$$H = \hbar v_F \boldsymbol{\sigma} \cdot \mathbf{k} + h_0 \sigma_z, \tag{7}$$

which provides a good approximation to the surface states of ferromagnetic topological insulators [46]. Note however that the LMR discussed in this work requires magnetic order. If magnetic order can be excluded, recent reports of LMR in topological insulators [8–11] must have a different origin.

For this simple model we can obtain explicit expressions for the Berry curvature, the orbital moment, and the spin moment along \hat{z} [42]. By substituting these results into the expression for $\sigma_{intr}^{(1)}$ and evaluating Eq. 6,

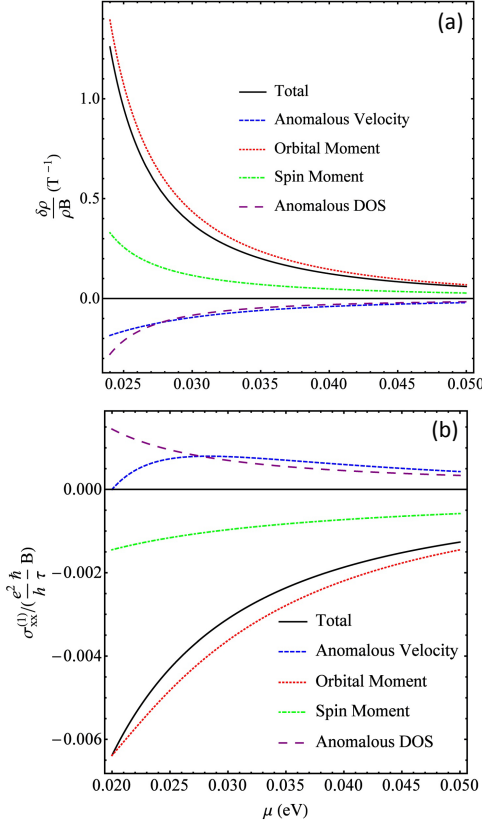


FIG. 2: (Color online) (a) and (b) Intrinsic contributions to LMR and to $\sigma_{xx}^{(1)}$ from different mechanisms *vs.* chemical potential μ at $T = 0$ K. $\hbar v_F = 4.1$ eV·Å for Bi₂Se₃ according to [47]. \hbar_0 is set to 20 meV, which is also the energy of the conduction band bottom. The g -factor is taken to be 50 [48].

the intrinsic LMR at zero temperature is obtained as

$$\begin{aligned} \frac{1}{B} \left[\frac{\rho_{xx}^{(1)}}{\rho_{xx}^{(0)}} \right]_{\pm} &= \pm \frac{1}{2} \frac{e \hbar v_F^2 \hbar_0}{(h_0^2 + \hbar^2 v_F^2 k_F^2)^{\frac{3}{2}}} \\ &\mp \left(\frac{h_0^2}{\hbar^2 v_F^2 k_F^2} + \frac{3}{2} \right) \frac{e \hbar v_F^2 \hbar_0}{(h_0^2 + \hbar^2 v_F^2 k_F^2)^{\frac{3}{2}}} \\ &- \frac{1}{2} \frac{g \mu_B \hbar_0}{h_0^2 + \hbar^2 v_F^2 k_F^2}. \end{aligned} \quad (8)$$

where the upper and lower signs refer to Fermi levels within the upper and lower bands plotted in Fig. 2 (a). The three terms on the right hand side correspond to the contributions from anomalous velocity/Berry curvature, orbital magnetic moment, and spin magnetic moment, respectively. Note that Eq. 6 and hence Eq. 8 is valid only when $\sigma_{xx}^{(0)}$ is smaller than the anomalous Hall conductivity, *i.e.*, the chemical potential is not very close to the band edges [42]. In this regime, the LMR is dominated by the contribution from the orbital magnetic moment, as shown in Fig. 2 (a), which has opposite signs for E_F

lying in different bands. It would be very interesting to see if this LMR sign change is observed experimentally.

When the chemical potential is close to the band edges of this 2D model so that $\sigma_{xx}^{(0)} < \sigma_{xy}^{(0)}$, instead of Eq. 6 we have

$$\frac{\rho_{xx}^{(1)}}{\rho_{xx}^{(0)}} \approx \frac{\sigma_{xx}^{(1)} \sigma_{xx}^{(0)}}{\sigma_{xy}^2}, \quad (9)$$

which means that at band edges where $\sigma_{xx}^{(0)} \rightarrow 0$ and $\sigma_{xy}^{(0)} \rightarrow e^2/2h$ the LMR still vanishes. (As we comment below, Boltzmann transport theory is not strictly valid in this regime, but its predictions are still suggestive.) However, it is interesting that $\sigma_{xx}^{(1)}$ (or $\rho_{xx}^{(1)}$) does not vanish with $\sigma_{xx}^{(0)}$, as shown in Fig. 2 (b). The nonzero contributions to $\sigma_{xx}^{(1)}$ at band edges, corresponding to the terms in Eq. 8 that are divergent when $k_F \rightarrow 0$, can be traced back to the last term in the 2nd line of Eq. 5. This term basically means that under a finite magnetic field both the band energy and the chemical potential are shifted due to the modifications to the equilibrium distribution function and to the Fermi volume by the magnetic field, and the conductivity needs to be calculated after the shift is accounted for [42]. This effect is more dramatic when the chemical potential is within the quantum anomalous Hall gap, where we can do an estimate of the change in electron density when the chemical potential is fixed:

$$\begin{aligned} \delta n &= - \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{eB}{\hbar} \Omega_- f_0 \\ &= \frac{\mathcal{C}}{2\pi l_B^2} = 1.2 \times 10^{10} \text{ cm}^{-2} B[\text{T}], \end{aligned} \quad (10)$$

where we have assumed that $T = 0$ K and that the density of states inside the gap vanishes. \mathcal{C} is the Chern number of the lower band, $l_B \equiv \sqrt{\frac{\hbar}{eB}}$ is the magnetic length, and B is in units of Tesla. Therefore to maintain charge neutrality the chemical potential must shift down, and if there are not enough in-gap states (including the quantum anomalous Hall edge state and any localized impurity states) that can provide such number of electrons, the chemical potential will simply jump to the valence band edge, from where the magnetoresistance increases linearly following Eq. 9. Conversely, if the magnetic field increases from zero along $-\mathbf{h}_0$ direction, the chemical potential jumps to the conduction band edge and the resistance also increases linearly from zero. Thus if the chemical potential initially lies inside the gap, the above mechanism will give rise to a magnetoresistance that is both linear and even in magnetic field, provided that the number of in-gap states is vanishingly small and that the quantum capacitance is much larger than the geometric capacitance for the capacitor formed by the 2D surface states and external gates [42].

We note that semiclassical Boltzmann transport theory is valid only when $\epsilon_F \tau / \hbar > 1$, where ϵ_F is the kinetic

energy of electrons at the Fermi surface. Beyond this regime quasiparticle scattering from different impurities is correlated and we must employ an appropriate quantum transport theory. For our current model this means that the longitudinal conductivity formulas derived above will hold when $\mu > h_0 + \hbar/\tau$. Nevertheless, the magnetic field induced chemical potential shift near the gap is still valid since it is an equilibrium property and not a result of Boltzmann transport theory.

Although the microscopic processes leading to LMR are closely related to those responsible for the AHE, it is not necessarily true that LMR is accompanied by the AHE. For example, in many antiferromagnetic materials the AHE is forbidden by structural symmetries [49]. A common example of such a case is the pyrochlore structure [50] with all-in-all-out magnetic order. However, it can be easily shown using the formalism presented above that the intrinsic contribution to LMR is still nonzero, even if one assumes the magnetic order is unchanged by the external fields. In general the symmetry requirements for a nonzero LMR in magnetic systems are less stringent than for the AHE.

HC and AHM were supported by ONR-N00014-14-1-0330 and Welch Foundation grant TBF1473. YG is supported by DOE (DE-FG03-02ER45958, Division of Materials Science and Engineering) and Welch Foundation (F-1255). DX is supported by AFOSR No. FA9550-14-1-0277. QN is supported by NBRPC (No.2012CB921300 and No.2013CB921900), and NSFC (No.91121004). The authors are grateful to Cui-Zu Chang, Dan Ralph, Yong Chen, and Fan Zhang for helpful discussions.

-
- [1] We exclude the linear magnetoresistance observed by Kapitza in Bi during the 1920's which is due to an open Fermi surface.
- [2] P. A. Penz and R. Bowers, Phys. Rev. **172**, 991 (1968).
- [3] B. K. Jones, Phys. Rev. **179**, 637 (1969).
- [4] R. Xu *et al.*, Nature **390**, 57 (1997).
- [5] J. Hu and T. F. Rosenbaum, Nature Mater. **7**, 697 (2008).
- [6] M. Csontos *et al.*, Phys. Rev. Lett. **95**, 227203 (2005).
- [7] A. L. Friedman *et al.*, Nano Lett. **10**, 3962 (2010).
- [8] D.-X. Qu *et al.*, Science **329**, 821 (2010).
- [9] H. Tang *et al.*, ACS Nano **5**, 7510 (2011).
- [10] X. Wang, Y. Du, S. Dou, and C. Zhang, Phys. Rev. Lett. **108**, 266806 (2012).
- [11] J. Tian, C. Chang, H. Cao, K. He, X. Ma, Q. Xue, and Y. P. Chen, Sci. Rep. **4**, 4859 (2014).
- [12] L. P. He, X. C. Hong, J. K. Dong, J. Pan, Z. Zhang, J. Zhang, and S. Y. Li, Phys. Rev. Lett. **113**, 246402 (2014).
- [13] T. Liang, Q. Gibson, M. N. Ali, M. Liu, R. J. Cava, and N. P. Ong, Nature Mater. **14**, 280-284 (2015).
- [14] J. Feng *et al.*, Phys. Rev. B **92**, 081306(R) (2015).
- [15] M. Novak, S. Sasaki, K. Segawa, and Y. Ando, Phys. Rev. B **91**, 041203(R) (2015).
- [16] A. Narayanan *et al.*, Phys. Rev. Lett. **114**, 117201 (2015).
- [17] C. Zhang *et al.*, arXiv:1502.00251
- [18] X. Huang *et al.*, Phys. Rev. X **5**, 031023 (2015).
- [19] A. A. Abrikosov, Phys. Rev. B **58**, 2788 (1998).
- [20] A. A. Abrikosov, Europhys. Lett. **49**, 789 (2000).
- [21] C. M. Wang and X. L. Lei, Phys. Rev. B **86**, 035442 (2012).
- [22] H. Nielsen and M. Ninomiya, Phys. Lett. B **130**, 389 (1983).
- [23] V. Aji, Phys. Rev. B **85**, 241101 (2012).
- [24] D. T. Son and B. Z. Spivak, Phys. Rev. B **88**, 104412 (2013).
- [25] A. A. Burkov, Phys. Rev. Lett. **113**, 247203 (2014).
- [26] C. Herring, J. Appl. Phys. **31**, 1939 (1960).
- [27] M. M. Parish and P. B. Littlewood, Nature **426**, 162 (2003).
- [28] N. A. Porter and C. H. Marrows, Sci. Rep. **2**, 565 (2012).
- [29] N. V. Kozlova *et al.*, Nat. Commun. **3**, 1097 (2012).
- [30] J. C. W. Song, G. Refael, and P. A. Lee, arXiv:1507.04730
- [31] L. Onsager, Phys. Rev. **37**, 405 (1931).
- [32] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Rev. Mod. Phys. **82**, 1539 (2010).
- [33] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
- [34] N. A. Sinitsyn, J. Phys.: Condens. Matter. **20**, 023201 (2008).
- [35] W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957).
- [36] Inversion symmetry is assumed so that the Berry curvature vanishes identically when time-reversal symmetry is not broken.
- [37] N. A. Sinitsyn, Q. Niu, and A. H. MacDonald, Phys. Rev. B **73**, 075318 (2006).
- [38] D. Xiao, J. Shi, and Q. Niu, Phys. Rev. Lett. **95**, 137204 (2005).
- [39] G. D. Mahan, J. Phys. F: Met. Phys. **14**, 941-962 (1984).
- [40] G. D. Mahan, J. Phys. F: Met. Phys. **13**, L257-L263 (1983).
- [41] A. A. Abrikosov, Sov. Phys. JETP **29**, 746 (1969).
- [42] Supplemental Material.
- [43] Yang Gao, Shengyuan A. Yang, and Qian Niu, Phys. Rev. Lett. **112**, 166601 (2014).
- [44] N. A. Sinitsyn, A. H. MacDonald, T. Jungwirth, V. K. Dugaev, and Jairo Sinova, Phys. Rev. B **75**, 045315 (2007).
- [45] J. Schliemann and D. Loss, Phys. Rev. B **68**, 165311 (2003).
- [46] C.-Z. Chang *et al.*, Science **340**, 167-170 (2013).
- [47] H. Zhang *et al.*, Nature Phys. **5**, 438 (2009).
- [48] J. G. Analytis, R. D. McDonald, S. C. Riggs, J.-H. Chu, G. S. Boebinger, and I. R. Fisher, Nature Phys. **6**, 960 (2010).
- [49] H. Chen, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. **112**, 017205 (2014).
- [50] T. Taguchi, Y. Oohara, H. Yoshizawa, N. Nagaosa, and Y. Tokura, Science **291**, 2573 (2001).